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Flow of a power law fluid in a rotating straight pipe.—I : Determination of flow field

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A steady flow of a power law fluid in a rotating straight pipe of circular cross-section is discussed in the present paper. The flow field consists of a primary axial flow and a secondary swirling flow in the meridian plane It is found that at low rotational speeds the swirl flow induced due to the rotation is weak and symmetrical about the plane of rotation in any cross-section As the rotation increases an almost shear free central region and a boundary layer type flow at the offside end of the cross-section develop Figures 1 to 6 depict these results

1. Introduction

Non-Newtonian fluids are found to be better heat transport media than the conventional Newtonian fluids as pointed out by Fraas & Ozisik (1965) Therefore the heat transfer phenomena in the flow of non-Newtonian fluids are of importance and need thorough investigation Many of the coolers use rotating devices and intrinsic cooling of rotating devices themselves are of great importance. Hence the study of flow of a non-Newtonian fluid in rotating pipes is of practical importance. The temperature distribution in any coolant is governed not only by the heat conduction but is very much dependent on the flow behaviour of the fluid as well as convective effects as seen from Raju & Rathna (1970) We first study the flow field of a power law fluid in a rotating straight pipe of circular cross-section in this part and consider the heat and mass transfer in a subsequent paper We have chosen power law fluids because amongst the class of non-Newtonian fluids they have the minimum number of empirical constants in their constitutive equation Also many realistic fluids can be approximated by power law fluids For instance, blood plasma can be treated as a power law fluid with flow behaviour index, $n = 0.92$ and many other high polymer solutions also behave similarly.

The corresponding flow in a rotating straight pipe for viscous incompressible Newtonian fluids for small angular speeds of rotation was first investigated by Barua (1954). He found that the flow field consists of a primary axial flow and a weak swirling secondary flow This resembles the flow generated in a curved pipe studied extensively by Dean (1927) and others. These investigations utilize mainly the perturbation technique Recently, a different approach to this problem has been made by Jones & Walters (1967). They determine the flow in the neighbourhood of the axis of rotation assuming that the meridional flow is shear free, following Dean & Hurst (1959). Because of this limitation, even though they obtain simpler equations and approximate flux, their solution is not valid away from the axis. Especially it does not satisfy the boundary conditions. But the heat transfer phenomena critically depend on the behaviour of the swirl flow near the boundary, as shown earlier by Raju & Rathna (1970), owing to the fact that the convective heat transfer is not negligible Consequently, the perturbation technique developed by Barua gives better results though restricted to small rotational speeds.

We determine the swirl flow induced by the rotation of the pipe in a primary pressure driven axial shear flow as a power series in rotational speed in the next section. We give a detailed discussion of the flow field later on. In part 2 we determine the temperature field and study the behaviour of Nusselt number with Dean's number as well as rotation.

FORMULATION OF THE PROBLEM

As given by Tomita (1959), the constitutive equation for a power law fluid is

$$T = -pI + \mu_p \Theta E \qquad \ldots \quad (2.1)$$

where T is the stress tensor, E is the rate of strain tensor, μ_p a constant and

$$\Theta = |E_{11}^2 + E_{22}^2 + E_{33}^2 + 2(E_{12}^2 + E_{23}^2 + E_{31}^2)|^{\frac{n-1}{2}}, \qquad \ldots \quad (2.2)$$

n being the flow behaviour index.

Consider the steady flow of the above fluid in a stright pipe rotating uniformly with an angular speed Ω. For simplicity of expressions, we refer the motion to a Cartesian frame of reference fixed in the pipe and rotating with it, such that the X-axis is along the axis of rotation and the Z-axis is along the axis of the pipe. The equations of continuity and momentum are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad \ldots \quad (2.3)$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial}{\partial X}(P/\rho) + \nu_p\left[\Theta\nabla^2 U + E_{xx}\frac{\partial \Theta}{\partial X} + E_{xy}\frac{\partial \Theta}{\partial Y}\right] \ldots \quad (2.4)$$

$$-2\Omega W + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{\partial}{\partial Y}(P/\rho) + \nu_p\left[\Theta\nabla^2 V + E_{xy}\frac{\partial \Theta}{\partial X} + E_{yy}\frac{\partial \Theta}{\partial Y}\right] \qquad \ldots \quad (2.4)$$

$$2\Omega V + U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} = -\frac{\partial}{\partial Z}(P/\rho) + \nu_p\left[\Theta\nabla^2 W + E_{xz}\frac{\partial \Theta}{\partial X} + E_{zy}\frac{\partial \Theta}{\partial Y}\right] \qquad \ldots \quad (2.6)$$

where $U(X, Y)$, $V(X, Y)$ and $W(X, Y)$ are the velocity components in the directions of X, Y and Z, respectively and $\nu_p = \mu_p/\rho$. The boundary conditions are

$$U = V = W = 0 \text{ on } F(X, Y) = 0, \quad \ldots \quad (2.7)$$

where $F(X, Y) = 0$ is the equation to the cross--section of the pipe. We introduce the stream function ψ by

$$U = -\frac{\partial \psi}{\partial Y}; \; V = \frac{\partial \psi}{\partial X} \quad \ldots \quad (2.8)$$

Substituting (2.8) in (2.4) to (2.6) and eliminating the pressure gradient term, we obtain

$$\left[\frac{\partial \psi}{\partial X} \cdot \frac{\partial}{\partial Y} - \frac{\partial \psi}{\partial Y} \cdot \frac{\partial}{\partial X}\right]\left(\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2}\right) - 2\Omega \frac{\partial W}{\partial X}$$

$$= \nu_p \left[\Theta \nabla^4 \psi + \frac{\partial \Theta}{\partial X}\left\{2\frac{\partial^3 \psi}{\partial X \partial Y^2} + 2\frac{\partial^3 \psi}{\partial X^3}\right\} + \frac{\partial \Theta}{\partial Y}\left\{2\frac{\partial^3 \psi}{\partial Y^3} + 2\frac{\partial^3 \psi}{\partial X^2 \partial Y}\right\}\right.$$

$$\left. + 4\frac{\partial^2 \Theta}{\partial X \partial Y} \cdot \frac{\partial^2 \psi}{\partial X \partial Y} + \left(\frac{\partial^2 \psi}{\partial X^2} - \frac{\partial^2 \psi}{\partial Y^2}\right)\left(\frac{\partial^2 \Theta}{\partial X^2} - \frac{\partial^2 \Theta}{\partial Y^2}\right)\right] \quad (2.9)$$

and

$$\nu_p \left[\Theta \nabla^2 W + \frac{\partial \Theta}{\partial X}\,\frac{\partial W}{\partial X} + \frac{\partial \Theta}{\partial Y} \cdot \frac{\partial W}{\partial Y}\right] = -C + \frac{\partial(\psi, W)}{\partial(X, Y)} + 2\Omega \frac{\partial \psi}{\partial X} \quad \ldots \quad (2.10)$$

where $C = \frac{1}{\rho}\frac{\partial P}{\partial z}$, is the given constant axial pressure gradient. To study the flow in a circular pipe of radius a we transform to cylindrical polar coordinates defined by

$$\left.\begin{aligned} Z &= R \sin \theta \\ Y &= R \cos \theta \\ Z &= Z \end{aligned}\right] \quad \ldots \quad (2.11)$$

Further, let

$$\begin{aligned} R = ar; \; \psi = \bar{U} a \psi; \quad W = \bar{W} w; \\ \Theta = \bar{W} a^{n-1} \Theta_1 \end{aligned} \quad \ldots \quad (2.12)$$

where $\bar{U}$ and $\bar{W}$ are characteristic velocities given by

$$\left.\begin{aligned} \bar{U} &= \frac{2\Omega a^{n+1}}{\nu_p}\left(\frac{C a^{n+1}}{\nu_p}\right)^{\frac{2-n}{n}} \\ \bar{W} &= \left(\frac{C a^{n+1}}{\nu_p}\right)^{\frac{1}{n}} \end{aligned}\right\} \quad \ldots \quad (2.13)$$

We expand the flow field in the form

$$\left.\begin{aligned} W &= w_0(r)+\frac{\bar{U}}{K\bar{W}}\, w_1(r,\theta)+\left(\frac{\bar{U}}{K\bar{W}}\right)^2 w_2(r,\theta)+\ldots \\ \frac{\psi}{K} &= \left(\frac{\bar{U}}{K\bar{W}}\right)\psi_1(r,\theta)+\left(\frac{\bar{U}}{K\bar{W}}\right)^2 \psi_2(r,\theta)+\ldots \end{aligned}\right\} \qquad \ldots \quad (2.14)$$

where the non-dimensional constant K is given by

$$K = \frac{Ca}{\bar{W}^2}. \qquad \ldots \quad (2.15)$$

Substituting (2.11) and (2.14) in (2.9) and (2.10) and comparing equal powers of $\frac{\bar{U}}{K\bar{W}}$, we get the equations satisfied by various order terms.

The differential equation for $w_0(r)$ is

$$\left[\left(\frac{dw_0}{dr}\right)^2\right]^{\frac{n-1}{2}}\left[n\,\frac{d^2w_0}{dr^2}+\frac{1}{r}\,\frac{dw_0}{dr}\right] = -1, \qquad \ldots \quad (2.16)$$

with the boundary condition

$$w_0 = 0, \text{ on } r = 1. \qquad \ldots \quad (2.17)$$

The well known solution of (2.16) is

$$w_0(r) = \frac{1}{2^{1/n}}\cdot\frac{n}{n+1}\cdot(1-r^{1+1/n}), \qquad \ldots \quad (2.18)$$

which is the solution of the pressure driven axial flow. We call this the primary motion of the problem. It is of interest to note that for $n = 1$, this reduces to Poiseuille flow.

The differential equation for ψ_1 is of the form

$$\frac{\partial^4\psi_1}{\partial r^4}+\frac{2n-1}{n^2}\cdot\frac{1}{r^3}\,\frac{\partial\psi_1}{\partial r}-\frac{2n-1}{n^2}\cdot\frac{1}{r^2}\,\frac{\partial^2\psi_1}{\partial r^2}+\frac{2n-1}{n}\cdot\frac{2}{r}\cdot\frac{\partial^3\psi_1}{\partial r^3}$$

$$-\frac{1}{n}\cdot\frac{2}{r^3}\cdot\frac{\partial^3\psi_1}{\partial r\partial\theta^2}+\frac{2}{r^2}\cdot\frac{\partial^4\psi_1}{\partial r^2\partial\theta^2}+\frac{1}{r^4}\,\frac{\partial^4\psi_1}{\partial\theta^4}+\frac{n^2+4n-1}{n^2}\,\frac{1}{r^4}\,\frac{\partial^2\psi_1}{\partial\theta^2}$$

$$= \sin\theta\,.\left[\frac{r}{2}\right]^{\frac{2-n}{n}}, \qquad \ldots \quad (2.19)$$

with the boundary conditions

$$\frac{\partial\psi_1}{\partial r} = \frac{\partial\psi_1}{\partial\theta} = 0 \quad \text{on} \quad r = 1, \qquad \ldots \quad (2.20)$$

Solving

$$\psi_1 = g(r)\sin\theta \qquad \ldots \quad (2.21)$$

where

$$g(r) = A'r + B'r^s + C'r^{2/n+3} \qquad \ldots \quad (2.22)$$

and

$$\left.\begin{aligned} A' &= C'\left[\frac{2+2n-n(s-1)}{n(s-1)}\right] \\ B' &= -C'\left[\frac{2+2n}{n(s-1)}\right] \\ C' &= \frac{1}{2^{(2-n)/n}} \cdot \frac{n^4}{4(n+1)(3n+1)(n^2+4n+1)} \\ s &= \frac{n+1}{2n} + \frac{\sqrt{(\sqrt{17}n-1)^2+2n(\sqrt{17}-1)}}{2n} \end{aligned}\right\} \qquad \ldots \quad (2.23)$$

With (2.18) and (2.21), it can be seen that w_1 satisfies the differential equattion,

$$nr^2\frac{\partial^2 w_1}{\partial r^2} + (2n-1)\,r\,\frac{\partial w_1}{\partial r} + \frac{\partial^2 w_1}{\partial \theta^2} = -\frac{1}{2^{(2-n)/n}}\left[A'r^{\frac{2+n}{n}} + B'r^{s+2/n} + C'r^{\frac{4}{n}+3}\right]\cos\theta. \qquad \ldots \quad (2.24)$$

the boundary condition being

$$w_1 = 0, \text{ on } r = 1. \qquad \ldots \quad (2.25)$$

The solution for w_1 is

$$w_1 = F(r)\cos\theta. \qquad \ldots \quad (2.26)$$

$F(r)$ is given by

$$F(r) = \left[\frac{r^{1/n}}{2^{(2-n)/n}}\right]\cdot\left[\frac{nA'}{2(n+1)^2}(1-r^{1+1/n}) + \frac{nB'(1-r^{s+1/n})}{(ns+1)(ns+2+n)} + \frac{nC'}{12(n+1)^2}(1-r^{3/n+3})\right] \qquad \ldots \quad (2.27)$$

where A', B', C' and s are given in (2.23).

This gives the solution up to the first order in the rotation parameters. We have carried out the analysis to the second order terms as well The expressions are complicated and lengthy. The order of magnitude of successive terms decreases vary rapidly and the principal features of the flow are given by first order term itself.

Discussion of the results

In order to study the solution (2 18), (2.21) and (2.26) we consider in detail three flow behaviour indices, $n = 0.8$ (psuedoplastic), $n = 1.0$ (Newtonian) and $n = 1.2$ (dilatant) which cover almost the whole range of power law fluids met with in practice. For instance blood plasma can be treated as a psuedoplastic liquid with $n = 0.92$ Secondly, we consider the cases of very small and fairly high rotational speeds. The velocity profiles are drawn for low rotational speeds in figures 1, 3 and 5, and for high rotational speeds in figures 2, 4 and 6 for $n = 0.8$, 1.0 and 1.2.

The secondary meridional flow is symmetrical about the plane of rotation in any cross-section At low rotational speeds the swirl flow generated is weak. But as the rotational speed increases an almost shearfree region develops in the central region with a boundary layer type flow at the offside end of the cross-section towards which the coriolis force acts The swirl flow detaches from the boundary and flows inside at the onside end of the cross-section. These conclusions are evident from the figures as well. This action of coriolis force is similar to the effect of centrifugal force due to curvature of a bent pipe, studied earlier by McConalogue & Srivastava (1968) for Newtonian fluids and by Raju & Rathna (1970) for power law fluids. Here this force is proportional to rotational speed and the pressure driven axial flow, and consequently not as strong as in the case of curvature effect, where it is proportional to square of the axial flow.

In all figures broken lines refer to ψ = constant and unbroken lines to W = constant.

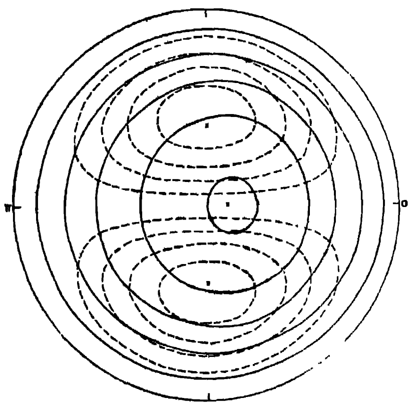

Figure 1. Secondary flow field for the Newtonian fluid at low rotational speeds ($n = 1.0$).

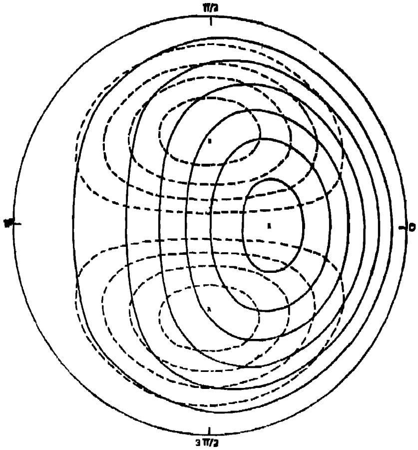

Figure 2. Secondary flow field for the Newtonian fluid at high rotationals speeds ($n = 1.0$).

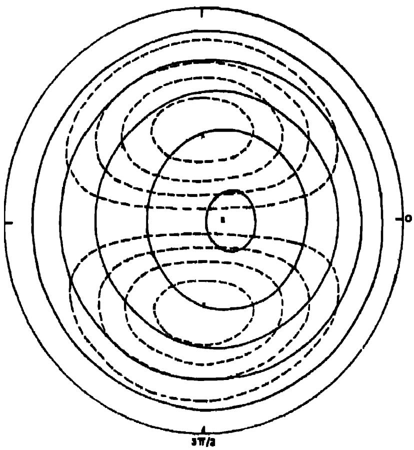

Figure 3. Secondary flow field for a dilatant fluid at low rotational speeds ($n = 1.2$).

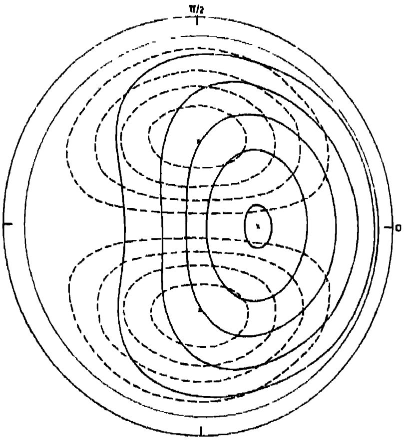

Figure 4. Secondary flow fluid for a dilatant fluid at high rotational speeds ($n = 1.2$).

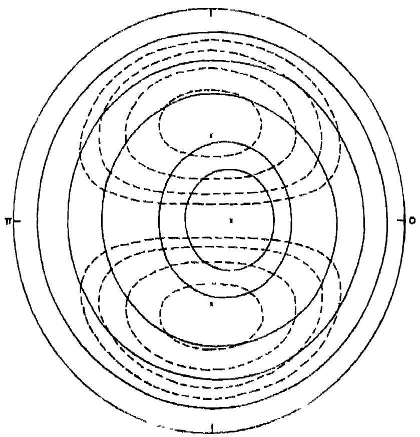

Figure 5. Secondary flow field for a pseudoplastic fluid at low rotational speeds ($n = 0.8$).

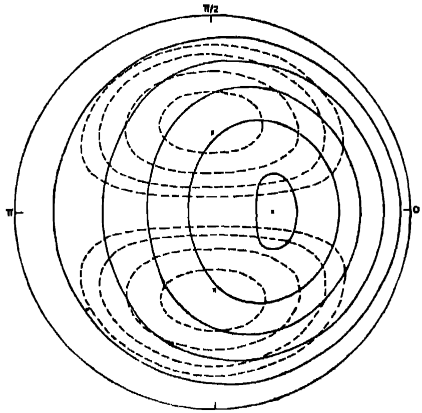

Figure 6. Secondary flow field for a pseudoplastic fluid at high rotational speeds ($n = 0.8$).

The following table gives the relative maxima of the axial speeds and positions where they occur.

TABLE 1

n	Low rotational speeds		High rotational speeds	
	W_{max}	Point of occuring	W_{max}	Point of occuring
0.8	0.4474	$r = 0.1$	0.5163	$r = 0.3$
1.0	0.5098	$r = 0.1$	0.6390	$r = 0.3$
1.2	0.5650	$r = 0.1$	0.7664	$r = 0.3$

It is seen that the W_{max} increases with the increasing values of n which may be explained by considering the fact that the psuedoplastic fluids ($n = 0.8$) sustain less strain and dilatant fluids ($n = 1.2$) sustain more strain. We notice that the point, where W_{max} occurs, shifts towards the offside as the rotation increases, but there is no variation in its occurrence with the flow behaviour index, n. This behaviour of flow field has great significance in the convective heat transfer pattern that develops.

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REFERENCES

Barua S. N. 1954 *Proc. Roy. Soc.* **227**A, 133.
Dean W. R. 1927 *Phil. Mag.* **4**, 208.
Dean W. R. & Hurst J. M. 1959 *Mathematika* **6**, 77.
Fraas A. P. & Ozısık M. N. 1965 *Heat Exchanger Design*, John Wiley and Sons Inc., New York.
Jones J. R. & Walters T. S. 1967 *ZAMP* **18**, 774.
McConalogue D. J & Srivastava R. S. 1968 *Proc. Roy. Soc.* **307**A, 37.
Raju Kanaka K. Rathna, S. L. 1970 *Jour. Ind. Inst. Sci.*, **52**, 34.
Tomita Y. 1959 *Bull. J. S. M. E.* **2**, 469.